Number QI-2004-01
in the Quantitative Imaging Group Technical Report Series

A short introduction to the Radon and Hough transforms and how they relate to each other

M. van Ginkel, C.L. Luengo Hendriks and L.J. van Vliet

michael@ph.tn.tudelft.nl

Delft University of Technology

Quantitative Imaging Group
Imaging Science & Technology Department
Faculty of Applied Science
Delft University of Technology
Lorentzweg 1, 2628 CJ Delft, The Netherlands
telephone: +31 15 2781416 fax: +31 15 2786740
http://www.ph.tn.tudelft.nl/

Abstract

Extraction of primitives, such as lines, edges and curves, is often a key step in an image analysis procedure. The most popular technique for curve detection is based on the Hough transform. The original formulation of the Hough transform is inherently discrete. It is therefore difficult to assess which properties are inherent to the transform-based technique and which are due to its discrete nature. As other authors have pointed out before, the Hough transform is closely related to the Radon transform, in fact equivalent, if one is not too pedantic about the original formulations of the two transforms. With this report we hope to once again stress this relationship. The Radon transform formalism has two advantages over the Hough formalism. It has a well-founded mathematical basis and, in our opinion, is more intuitive as well.

Contents

1 Introduction

One of the first stages in image analysis is the extraction of primitives, such as lines, edges, curves or simple textures, from an image. In this paper we focus on curve detection, or more precisely, *shape* detection. In three- and higher-dimensional spaces, manifolds (N-dimensional), such as a spherical membrane, are as interesting as curves (one-dimensional). In general we are interested in a given *family of shapes*. Our assumption is that the members of this family can be described by a set of parameters. The task, then, is to find the parameters corresponding to the best fitting member of the family of shapes. The standard method for detecting parameterized shapes is based on a family of transformations, which includes the Radon [24] and Hough [11] transforms. The purpose of this report is to give a brief introduction to these transforms and to stress the relationship between them. Before moving on to discuss the transforms themselves, we first examine how shapes are described.

1.1 Shape representation using generalized functions

Before proceeding, we introduce the following notation:

$\mathbf{x}$	The spatial coordinates
$I(\mathbf{x})$	The D-dimensional image containing the N-dimensional shapes
$\mathbf{p}$	The vector containing the parameters of the curve. Often a subset of the parameters specifies the location of the shape. It is, therefore, sometimes convenient to write $\mathbf{p} = \{\mathbf{q}, \mathbf{x}_o\}$, with $\mathbf{x}_o$ the location of the shape (the center of the sphere), and $\mathbf{q}$ the remainder of the parameters (the radius of the sphere).
$c(\mathbf{p})$	A member of a class of shapes described by the parameter vector $\mathbf{p}$.
$\mathbf{c}(\mathbf{s}; \mathbf{p})$	The coordinates of a point belonging to the shape $c(\mathbf{p})$. The coordinates $\mathbf{s}$ allow us to specify a specific point on the shape.
$\mathcal{C}(\mathbf{x}; \mathbf{p})$	A set of constraint functions that together define the shape. The number of constraint functions depends on the dimensionality of the shape: $D - N$ constraints are necessary to describe a N-dimensional shape. For a point that lies on the shape, all the constraint functions evaluate to zero: $\mathcal{C}_i(\mathbf{x}; \mathbf{p}) = 0$ for all i.
$C(\mathbf{p}, \mathbf{x})$	A kernel, also called template, that represents the shape given by $\mathbf{p}$ as an image with spatial coordinates $\mathbf{x}$. We can model the image I as a sum of several of these templates.

Shapes can be described in different ways. The notation $\mathbf{c}(\mathbf{s}; \mathbf{p})$ represents the shape. For a circle in 2D centered at $\mathbf{x}_o$ and with radius r this becomes

$$\mathbf{c}(\phi; \{r, \mathbf{x}_o\}) = \mathbf{x}_o + r \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \tag{1}$$

with ϕ a free coordinate letting us specify an arbitrary point on the circle. Alternatively, a shape can be defined through the specification of a constraint; this is known as the implicit representation. In the case of a circle:

$$\mathcal{C}(\mathbf{x}; \{r, \mathbf{x}_o\}) = 0 \;\text{ with }\; \mathcal{C}(\mathbf{x}; \{r, \mathbf{x}_o\}) = ||\mathbf{x} - \mathbf{x}_o|| - r. \tag{2}$$

Now recall that the shapes we are looking for are embedded in an image and not directly available as a set of points. This means that standard results from differential geometry, such as the expression for the curvature of a plane curve [31]

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \tag{3}$$

are not directly applicable. In this example, the curvature of a curve embedded in an image in the form of an isophote can still be obtained through the well-known result for the isophote curvature [15,33,16].

In general, however, it may be beneficial to make the embedding of the shapes explicit. The basic ingredients for such a description are the constraint-based description and the Dirac delta function. The theoretical basis for this description can be found in Gel'fand et al. [9, Chapter III, Section 1, "Generalized Functions Concentrated on Smooth Manifolds of Lower Dimension"], who give a very lucid account of this subject matter. It is not our intention to give a complete exposition of this material; we will merely touch upon the essentials.

Consider an N-dimensional shape in D-dimensional space. At any point on the shape we can span a local coordinate system. We will denote the local coordinate vector by $\mathbf{u}$. The first N coordinates $u_{1 \ldots N}$ span the subspace in which the shape lies and the $D - N$ remaining coordinates $u_{(N+1) \ldots D}$ span the subspace normal to the shape. In fact, these last coordinates act as constraint functions: we can set $\mathcal{C}_i = u_{i+N}$. We can describe the infinitesimal neighborhood I^n by

$$I^n(\mathbf{u}) = \delta(u_{(N+1)}, \ldots, u_D). \tag{4}$$

If we choose an orthogonal coordinate system, then this reduces to

$$I^n(\mathbf{u}) = \prod_{i=N+1}^{D} \delta(u_i). \tag{5}$$

The scaling of the u_i should be such that they correspond to the Euclidean distance to the shape. This ensures that a) the points on the shape contribute equally in an integral and b) the overall scale is such that the $D-$dimensional volume integral over the image (assuming there is only the one shape) yields the N-dimensional hyper-volume of the shape. If the constraint functions are chosen according to these principles, we write

$$I(\mathbf{x}) = \delta(\mathcal{C}(\mathbf{x}; \mathbf{p})). \tag{6}$$

Let us examine two simple examples in three-dimensional space. The $x - y$ plane is described by the constraint $z = 0$. Therefore,

$$I_{[x\,-\,y \text{ plane}]}(x, y, z) = \delta(z), \tag{7}$$

represents a plane in the $x - y$ plane. A line along the x-axis is described by the constraint $y = 0$ and $z = 0$:

$$I_{[\text{line along } x]}(x, y, z) = \delta(y)\delta(z). \tag{8}$$

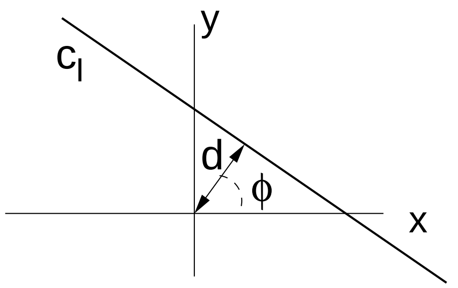

Fig. 1. The *normal* parameterization of a line. The parameters are the distance d from the line to the origin, through the normal of the line that intersects the origin, and the angle ϕ between that same normal and the x-axis, as indicated in the diagram.

2 The Radon transform

The Radon transform is named after J. Radon who showed how to describe a function in terms of its (integral) projections [24]. The mapping from the function onto the projections is the Radon transform. The inverse Radon transform corresponds to the reconstruction of the function from the projections. The original formulation of the Radon transform is as follows:

$$\mathcal{R}\{I\}(d,\phi) = \int_{\mathbb{R}} I(d\cos\phi - s\sin\phi, d\sin\phi + s\cos\phi)ds, \tag{9}$$

with the projection along the lines $c_l(d,\phi)$ with the parameterization as given in Figure 1. Within the realm of image analysis, the Radon transform is mostly known for its role in computed tomography. It is used to model the process of acquiring projections of the original object using X-rays. Given the projection data, the inverse Radon transform, in whatever form (e.g. backprojection), can be applied to reconstruct the original object.

The Radon transform can also be used for shape detection. We reformulate the Radon transform:

$$\mathcal{R}\{I\}(d,\phi) = \int\limits_{(x,y) \text{ on } c_l(d,\phi)} I(x,y)\,dx\,dy = \int\limits_{\mathbb{R}^R} I(x,y)\delta(x\cos\phi + y\sin\phi - d)\,dx\,dy. \tag{10}$$

It is now trivial to generalize the Radon transform to arbitrary shapes $c(\mathbf{p})$. We give three equivalent formulations, leaving it to the reader to decide which is the easiest to interpret:

$$\mathcal{R}_{c(\mathbf{p})}\{I\}(\mathbf{p}) = \int\limits_{\mathbf{x} \text{ on } c(\mathbf{p})} I(\mathbf{x})\,d\mathbf{x} = \int\limits_{\mathbb{R}^N} I\big(\mathbf{c}(\mathbf{s};\mathbf{p})\big)\,||\frac{\partial \mathbf{c}}{\partial \mathbf{s}}||\,d\mathbf{s} = \int\limits_{\mathbb{R}^D} I(\mathbf{x})\delta(\mathcal{C}(\mathbf{x};\mathbf{p}))\,d\mathbf{x}. \tag{11}$$

The third formulation expresses the Radon transform as a volume integral, a form that is particularly practical in image analysis. The mathematical properties of this generalized form of the Radon transform have been extensively studied in [8].

Now imagine that there is a shape in the image with parameter set $\mathbf{a}$. When $\mathbf{p} \neq \mathbf{a}$, the Radon transform will evaluate to some finite number which is proportional to the number of intersections between the shapes $c(\mathbf{p})$ and $c(\mathbf{a})$, as illustrated in Figure 2. However, when $\mathbf{p} = \mathbf{a}$, the Radon transform yields a large response (a peak in the parameter space). This response is proportional to the N-dimensional hyper-volume of the shape. We can now interpret the Radon transform as follows: it provides a mapping from image space to a *parameter space* spanned by the parameters $\mathbf{p}$. The function created in this parameter space, $P(\mathbf{p})$, contains peaks for those $\mathbf{p}$ for which the corresponding shape $c(\mathbf{p})$ is present in the image. Shape detection is reduced to the simpler problem of peak detection.

The third formulation of the Radon transform in equation (11) demonstrates an important reason for using generalized functions. In this notation, we can recognize the form of a linear integral operator[1] $\mathcal{L}_K$ with kernel C:

$$(\mathcal{L}_C I)(\mathbf{p}) = \int_{\mathbb{R}^D} C(\mathbf{p}, \mathbf{x}) I(\mathbf{x})\, d\mathbf{x}. \tag{12}$$

Therefore, if we allow the kernel C to be a generalized function, the Radon transform can be treated as any other linear transformation. In fact, using generalised functions, the identity operator (using the Dirac delta) as well as differential and integral operators (using derivatives and primitives of the Dirac delta) can be described in integral form. Dirac [6] introduced these in order to develop a continuous equivalent to matrix algebra in his work on quantum mechanics.

In case of a Radon transform, the kernel C is of the form: $C(\mathbf{p}, \mathbf{x}) = \delta(\mathcal{C}(\mathbf{x}; \mathbf{p}))$. In terms of shape detection, the role of the operator $\mathcal{L}_C$ is to compute the match (the inner product) between the image and a template C for a given parameter set $\mathbf{p}$. Here we see the connection between the Radon transform and template matching.

Often, the parameters $\mathbf{p}$ consist of the position of the shape $\mathbf{x}_o$ and the actual shape parameters $\mathbf{q}$. In this case the kernel has a special (shift-invariant) structure:

$$C(\{\mathbf{q}, \mathbf{x}_o\}, \mathbf{x}) = C(\{\mathbf{q}, \mathbf{x}_o + \mathbf{d}\}, \mathbf{x} + \mathbf{d}) \quad \text{for any } \mathbf{d}. \tag{13}$$

The operator $\mathcal{L}_C$ now reduces to a set of convolutions:

$$(\mathcal{L}_C I)(\mathbf{q}, \mathbf{x}_o) = \big(K_C(\mathbf{q}) *_{\mathbf{x}} I\big)(\mathbf{x}_o) \quad \text{with} \quad K_C(\mathbf{q}, \mathbf{x}) = C(\{\mathbf{q}, \mathbf{x}\}, \mathbf{0}). \tag{14}$$

This implies a large speed-up: using the convolution property of the Fourier transform, each convolution reduces to a multiplication in the Fourier domain.

Use of the Radon transform for shape detection dates back to 1965 [4]. The technique these authors describe is essentially a Radon transform. Rosenfeld [25] describes this technique (for straight lines) in Section 8.4.e "Coordinate Conversion". Neither [4] nor [25] identify this technique as the Radon transform.

3 The Hough transform

Rosenfeld [25] describes *two* techniques for curve detection: the first corresponds, as stated in the previous section, to the Radon transform. The second is a transform due to Hough [11], which through work by early adopters [7,13,20,26] has become very

[1] This is known as a Fredholm operator.

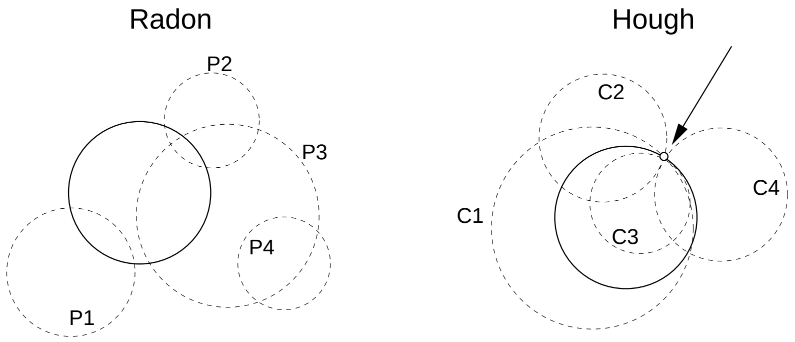

Fig. 2. The Radon and Hough transforms explained. Left: the Radon transform. Integrating the intensity values along each of the candidate curves P1–P4 yields small numbers. Only if a candidate curve happens to fully coincide with a curve in the image (the solid black circle), will the integral yield a large response. Right: the Hough transform. The indicated point can be accounted for by the presence of any of the indicated candidate curves C1–C4. If we consider all the points in the image in turn, we get many curves that account for individual points, but only candidate curves that correspond to a true curve (in this case the solid circle) will account for all of the points.

popular. Many authors have since noted that the Radon and Hough transforms are very closely related.

The Hough transform was originally defined to detect straight lines in black and white images, and seemingly inherently discrete. As it is trivial to generalize the Hough transform to other shapes and grey-value images, we describe it in this extended form. We set up an N-dimensional accumulator array, each dimension corresponding to one of the parameters of the shape looked for. Each element of this array contains the number of votes for the presence of a shape with the parameters corresponding to that element. The votes themselves are obtained as follows. Consider each point in the input image in turn. Now we consider which shapes this point, with grey value g, could potentially be a member of, see Figure 2. We increment the vote for each of these shapes with g. Of course, if a shape with parameters $\mathbf{p}$ is present in the image, all of the pixels that are part of it will vote for it, yielding a large peak in the accumulator array. The Hough transform, like the Radon transform, is a mapping from *image space* to a *parameter space*.

What is the relation between the Hough transform and the Radon transform? The Radon transform is a mapping. A mapping can be approached from two points of view. In the first we consider how a data point in the destination space is obtained from the data in the source space: the *reading paradigm*. This is how the Radon transform is usually treated. The alternative is to consider how a data point in the source space maps onto data points in the destination space: *the writing paradigm*. This is exactly what the Hough transform does, albeit in a discrete setting. According to this picture the Hough transform is a *discretisation* of the (continuous) Radon transform.

The mathematical formalism for the two methods is equivalent and given by equation (12) with kernel functions of the form $\delta(\mathcal{C}(\mathbf{x};\mathbf{p}))$. The mathematical formalism allow two different computational interpretations:

Reading paradigm (Radon): For each $\mathbf{p}$, collect all the values of $I(\mathbf{x})$, apply the template weights $K(\mathbf{x};\mathbf{p})$, and sum everything.

Writing paradigm (Hough): Initialize the entire function $P(\mathbf{p})$ to zero. For each point $\mathbf{x}$ in the input image determine its contribution, weighted with $K(\mathbf{x};\mathbf{p})$, to each of the points in $P(\mathbf{p})$ and update $P(\mathbf{p})$.

The difference in interpretation can be beneficial: if the input data is sparse, the Hough paradigm offers an immediate reduction in computation time. Vice versa, if we are interested in only a view points in parameter space, then the Radon paradigm is to be preferred.

With their respective roles clear, we benefit from both the mathematical formalism of the Radon transform and the literature on practical issues and extensions surrounding the Hough transform that have been published the last couple of decades: error analysis [26,27,28,32,21,14,23,1], reduction of the computational complexity (references in [19]), extensions [3,2,10] (to e.g. other shapes or use of additional information, such as orientation), and choosing the appropriate parameterization [7,34,35]. An extensive survey of the Hough transform literature up to 1988 is given in [12].

The equivalence of the Radon transform, Hough transform and template matching has been discussed by several authors. Stockman and Agrawala [30], and Sklansky [29] have used arguments similar to those above to demonstrate the equivalence of the Hough transform and templace matching. The formulation by Gel'fand et al. [8] of the Radon transform in terms of the Dirac delta function is in fact a form of template matching. Deans [5] was the first to establish the equivalence of the Radon and Hough transforms, as well as the first to bring the work of Gel'fand et al. to the attention of the field of image analysis.

Finally, Princen et al. [22] have given a continuous formulation of the Hough transform, using an interesting approach that is perhaps more in the spirit of the "Hough frame of mind". At the basis for their formulation are the constraint functions $\mathcal{C}$. For any given point $\mathbf{x}$ in the input space, the constraint(s) $\mathcal{C}(\mathbf{x};\mathbf{p})$ trace out a manifold in the parameter space spanned by the parameters $\mathbf{p}$. Multiple points $\mathbf{x}$ give rise to multiple manifolds. These will intersect each other at the point $\mathbf{p}_0$ in parameter space, thus identifying the curve. The mathematical formulation of this principle is given by the familiar relation (12), thus unifying this approach with the others given above. Their claim that the Radon and Hough transforms are not equivalent, seems to be based on a) comparing the continuous Radon transform to the original discrete Hough transform, rather than comparing the continuous definitions, and b) not recognising that the Radon transform can be written in the form of equation (12), despite using the Dirac delta in their own formulation of the Hough transform.

4 Discussion

The insight that the Radon transform, Hough transform are equivalent and that both can be seen as a form of template matching is not new nor unknown. Yet, most of the literature on shape detection is based on the Hough formalism. Many authors do acknowledge the link between the two, but it is often presented as just an interesting titbit. There are textbooks that do discuss both transformations, but usually with the Radon transform in the context of computed tomography, and the Hough transform in that of shape detection.

It is our hope that this document acts as a reminder to some and brings the Radon/Hough unity to the attention of others. We feel that the Radon transform formalism provides an intuitive understanding and a sound mathematical basis, whereas the Hough transform can be seen as a clever discretisation for a particular class of input data. Also, due to the large interest in the Hough transform over the last few decades, it boasts a large body of literature containing both theoretical and practical results. Awareness of both the Radon transform and the Hough transform can therefore be particularly fruitful.

On a final note, the first steps towards the Radon formalism overtaking the Hough formalism may already have been taken: Leavers, author of a book on the Hough transform [17], has recently published an interesting paper on shape descriptors based on the Radon transform [18].

References

1. A.S. Aguado, E. Montiel, and M.S. Nixon. Bias error analysis of the generalised Hough transform. *Journal of Mathematical Imaging and Vision*, 12(1):25–42, 2000.
2. A.S. Aguado, M.E. Montiel, and M.S. Nixon. Ellipse detection via gradient direction in the Hough transform. In *Fifth International Conference on Image Processing and its Applications*, pages 375–378, 4–6 July 1995.
3. D.H. Ballard. Generalizing the Hough transform to detect arbitrary shapes. *Pattern Recognition*, 13(2):111–122, 1981.
4. M.J. Bazin and J.W. Benoit. Off-line global approach to pattern recognition for bubble chamber pictures. *IEEE Transactions on Nuclear Science*, 12:291–295, August 1965.
5. S.R. Deans. Hough transform from the Radon transform. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 3(2):185–188, March 1981.
6. P. Dirac. The physical interpretation of the quantum dynamics. *Proceedings of the Royal Society of London*, A113:621–641, 1927.
7. R.O. Duda and P.E. Hart. Use of the Hough transformation to detect lines and curves in pictures. *Communications of the ACM*, 15(1):11–15, 1972.
8. I.M. Gel'fand, M.I. Graev, and N.Ya. Vilenkin. *Generalized Functions. Volume 5, Integral Geometry and Representation Theory*. Academic Press, 1966.
9. I.M. Gel'fand and G.E. Shilov. *Generalized Functions. Volume 1, Properties and Operations*. Academic Press, 1964.
10. M. van Ginkel. *Image Analysis using Orientation Space based on Steerable Filters*. PhD thesis, Delft University of Technology, Delft, The Netherlands, 2002. digital version available:
http://www.ph.tn.tudelft.nl/PHDTheses/MvGinkel/thesis_vanginkel.html.
11. P.V.C. Hough. Method and means for recognizing complex patterns. US patent nr. 3069654, 1962.
12. J. Illingworth and J. Kittler. A survey of the Hough transform. *Computer Vision, Graphics and Image Processing*, 44(1):87–116, 1988.
13. C. Kimme, D. Ballard, and J. Sklansky. Finding circles by an array of accumulators. *Communications of the ACM*, 18(2):120–122, February 1975.

14. N. Kiryati and A.M. Bruckstein. What 's in a set of points? *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 14(4):496–500, April 1992.
15. L. Kitchen and A. Rosenfeld. Gray-level corner detection. *Pattern Recognition Letters*, 1(2):95–102, 1982.
16. J.J. Koenderink and W. Richards. Two-dimensional curvature operators. *Journal of the Optical Society of America A*, 5(7):1136–1141, July 1988.
17. V.F. Leavers. *Shape Detection in Computer Vision Using the Hough Transform*. Springer-Verlag, London, 1992.
18. V.F. Leavers. Use of the two-dimensional Radon transform to generate a taxonomy of shape for the characterization of abrasive powder particles. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(12):1411–1423, December 2000.
19. C.L. Luengo Hendriks, M. van Ginkel, P. Verbeek, and L.J. van Vliet. The generalized radon transform: Sampling, accuracy and memory considerations. *to be submitted*, 2004.
20. P.M. Merlin and D.J. Farber. A parallel mechanism for detecting curves in pictures. *IEEE Transactions on Computers*, 24:96–98, January 1975.
21. W. Niblack and D. Petkovic. On improving the accuracy of the Hough transform: Theory, simulations, and experiments. In *Proceedings of the IEEE Computer Society Conference CVPR (Ann-Arbor)*, pages 574–579, June 1988.
22. J. Princen, J. Illingworth, and J. Kittler. A formal definition of the Hough transform: Properties and relationships. *Journal of Mathematical Imaging and Vision*, 1:153–168, 1992.
23. J. Princen, J. Illingworth, and J. Kittler. Hypothesis testing: A framework for analyzing and optimizing Hough transform performance. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 16(4):329–341, April 1994.
24. J. Radon. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Berichte Sächsische Akademie der Wissenschaften, Leipzig, Mathematisch-Physikalische Klasse*, 69:262–277, 1917.
25. A. Rosenfeld. *Picture Processing by Computer*. Academic Press, 1969.
26. S.D. Shapiro. Transformations for the computer detection of curves in noisy pictures. *Computer Graphics and Image Processing*, 4:328–338, 1975.
27. S.D. Shapiro. Feature space transforms for curve detection. *Pattern Recognition*, 10:129–143, 1978.
28. S.D. Shapiro. Properties of transforms for the detection of curves in noisy pictures. *Computer Graphics and Image Processing*, 8:219–236, 1978.
29. J. Sklansky. On the Hough technique for curve detection. *IEEE Transactions on Computers*, 27(10):923–926, October 1978.
30. G.C. Stockman and A.K. Agrawala. Equivalence of Hough curve detection to template matching. *Communications of the ACM*, 20(11):820–822, 1977.
31. J.J. Stoker. *Differential Geometry*. John Wiley and Sons, Inc, 1969.
32. T.M. van Veen and F.C.A. Groen. Discretization error in the Hough transform. *Pattern Recognition*, 14:137–145, 1981.
33. P.W. Verbeek. A class of sampling-error free measures in oversampled band-limited images. *Pattern Recognition Letters*, 3:287–292, 1985.
34. C.-F. Westin and H. Knutsson. The Möbius strip parameterization for line extraction. In *Computer Vision – ECCV92. Proceedings, 1992*, volume 588, pages 33–38. Springer-Verlag, 1992.

35. S.Y. Yuen and C.H. Ma. An investigation of the nature of parameterization for the Hough transform. *Pattern Recognition*, 30(6):1009–1040, 1997.